Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
International Journal of Solids and Structures 42 (2005) 4099–4114
INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES
www.elsevier.com/locate/ijsolstr

On linking n-dimensional anisotropic and isotropic Green's functions for infinite space, half-space, bimaterial, and multilayer for conduction

Tungyang Chen *, Hsin-Yi Kuo

Department of Civil Engineering, National Cheng Kung University, Tainan 70101, Taiwan

Received 30 September 2004; received in revised form 18 December 2004

Abstract

We establish exact mathematical links between the n-dimensional anisotropic and isotropic Green's functions for diffusion phenomena for an infinite space, a half-space, a bimaterial and a multilayered space. The purpose of this work is not to attempt to present a solution procedure, but to focus on the general conditions and situations in which the anisotropic physical problems can be directly linked with the Green's functions of a similar configuration with isotropic constituents. We show that, for Green's functions of an infinite and a half-space and for all two-dimensional configurations, the exact correspondences between the anisotropic and isotropic ones can always be established without any regard to the constituent conductivities or any other information. And thus knowing the isotropic Green's functions will readily provide explicit expressions for anisotropic Green's functions upon back transformation. For three- and higher-dimensional bimaterials and layered spaces, the correspondence can also be found but the constituent conductivities need to satisfy further algebraic constraints. When these constraints are fully satisfied, then the anisotropic Green's functions can also be obtained from those of the isotropic ones, or at least in principle.

1. Introduction

Finding Green's function for certain physical phenomena is one of the fundamental subjects in mathematical physics. For example, Green's functions for an unbounded space could serve as a theoretical basis in boundary integral formulation, in finding field solutions for boundary valued problems through

* Corresponding author. Tel.: +886 6 2757575 63121; fax: +886 6 2358542.
E-mail address: tchen@mail.ncku.edu.tw (T. Chen).

0020-7683/$ - see front matter © 2005 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijsolstr.2004.12.016

superpositions, and in estimating the effective properties of heterogeneous media. Classical fundamental solutions have been known for decades, such as in conduction, elasticity, poroelasticity and piezoelectricity (see for instance Ting, 1996; Norris, 1994; Chen, 1993 and the references contained therein). Among various physical phenomena, the mathematics of (two-dimensional) steady-state heat conduction (equivalent to anti-plane shear deformation in elastic cylindrical bodies) is probably the simplest. And thus it sometimes permits us to explore the Green's functions in a generally anisotropic setting and in a higher-dimensional space. For certain non-homogeneous configurations, it is also possible to derive the Green's functions analytically. For example, in a recent study (Kuo and Chen, 2005), they have analytically derived the Green's functions in conduction for an exponentially graded anisotropic solid in simple, closed forms.

In this work, we are concerned with the n-dimensional anisotropic Green's functions in conduction for an unbounded space, a half-space, a bimaterial and a multilayered space. The purpose of this work is not to attempt to present a solution procedure for these boundary valued problems, but to focus on the general conditions and situations in which the physical problems for anisotropic solids can be directly linked with the corresponding Green's functions of a similar configuration but with isotropic constituents. When the latter solutions are available or can be resolved, one can then obtain the anisotropic Green's functions upon back transformation, or at least in principle. In the formulation, we make use of an affine coordinate transformation (Sokolnikoff, 1956; Milton, 2002) for all configurations. In related subjects, the idea of an affine transformation was employed by Sokolnikoff (1956, §51) and Horgan and Miller (1994) in showing how the torsion problem for an orthotropic shaft can be reduced to that of an isotropic one, and by Milton (2002) in estimating the effective conductivities of an ellipsoidal assemblage. In the present context, for each constituent region in a bimaterial or in a layered space, we introduce a particular transformation matrix relevant to its conductivity tensor, and possibly along with a constant shifting vector. The field in each region is then governed by a standard Laplace equation, with a certain modification on the boundary term and/or the interfacial continuity conditions. For Green's functions for a half-space, three different kinds of homogeneous boundary conditions in conductions are considered (Carslaw and Jaeger, 1959). For the first-kind boundary condition the temperature is set equal to zero along the boundary; for the second-kind the normal component of the heat flux is taken to be zero; for the third kind a linear combination of the temperature and the normal component of the heat flux is set to be zero. Perfect bonding conditions are assumed to prevail at interfaces between any two adjacent regions for a layered medium or in a bimaterial. We show that the interfacial continuity conditions and the boundary term under the affine transformation remain intact. But for three- and higher-dimensional spaces the interfacial points may separate. To ensure the points meet perfectly, additional algebraic constraints must be fulfilled among the phase conductivities. This poses additional constraints to set up the linkage between the anisotropic and isotropic Green's functions. Specifically, we show that, for an infinite and a half-space and for all two-dimensional configurations, the exact correspondence between the anisotropic and isotropic Green's functions can always be constructed without any regard to the phase conductivities. And thus knowing the isotropic Green's functions will readily provide explicit expressions for anisotropic Green's functions upon back transformation. For a three- or higher-dimensional bimaterial and layered space, the correspondence can only be found if the constituent conductivities satisfy further algebraic constraints. When these constraints are fully satisfied, then the anisotropic Green's functions can be exactly expressed in terms of those with isotropic constituents. Interestingly, given the same dimensionality, it turns out that the constraint conditions for the configurations of a bimaterial and a multilayered medium have the same forms.

The plan of the paper is as follows. In Section 2, the concept of an affine coordinate transformation is introduced in the formulation and the Green's function in conduction for an n-dimensional anisotropic space is reconstructed. Exact expressions for the transformation matrix will be discussed in Appendix A. Section 3 considers the Green's functions for a half-space with three different kinds of homogeneous boundary conditions. Sections 4 and 5 examine the Green's functions for a bimaterial and a layered medium, respectively. For reference, the isotropic Green's functions of an infinite space, a half-space, and a bimaterial are recorded in Appendices B and C, which will be employed to construct the anisotropic Green's functions directly.

2. Green's function for an infinite space

We first consider the Green's function for steady-state conduction in an n-dimensional unbounded space. The conductivity tensor $\mathbf{k}_n$ is generally anisotropic satisfying the symmetry condition $k_{ij} = k_{ji}$. Here the subscript n in $\mathbf{k}_n$ designates the dimensionality of the space, e.g., $n = 2$ for a two-dimensional plane and $n = 3$ for a three-dimensional space. Under steady-state conditions, the field equation for the temperature field subjected to a point heat source at $\mathbf{x}_0$ has the form

$$k_{ij}\frac{\partial^2 G}{\partial x_i \partial x_j} + \delta(\mathbf{x} - \mathbf{x}_0) = 0. \tag{2.1}$$

Here $\delta(\mathbf{x} - \mathbf{x}_0)$ is the Dirac delta function and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is a position vector in reference to a fixed Cartesian coordinate. Note that throughout the paper sums over repeated indices, $i, j = 1, 2, \ldots, n$, are implied. The unknown function G, known as the Green's function, is the temperature field at the point $\mathbf{x}$ due to a point heat source applied at $\mathbf{x}_0$. This field equation (2.1) is the standard oblique-derivative boundary-value problem for a second-order partial differential equation. To resolve (2.1), we introduce an affine coordinate transformation (Sokolnikoff, 1956; Milton, 2002)

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \tag{2.2}$$

where the transformation matrix $\mathbf{A}$ does not depend on $\mathbf{x}$. Using the chain rule of differentiations, we have

$$\frac{\partial^2 G}{\partial x_i \partial x_j} = A_{ki}A_{lj}\frac{\partial^2 G}{\partial x'_k \partial x'_l}. \tag{2.3}$$

Since $\delta(\mathbf{x} - \mathbf{x}_0) = |J|\,\delta(\mathbf{x}' - \mathbf{x}'_0)$ (DeSanto, 1992) with $|J|$ being the absolute value of the Jacobian defined by $|J| \equiv |\partial x'_i/\partial x_j| = \det\mathbf{A}$, Eq. (2.1) can be recast as

$$A_{ki}k_{ij}A_{lj}\frac{\partial^2 G}{\partial x'_k \partial x'_l} + |J|\,\delta(\mathbf{x}' - \mathbf{x}'_0) = 0. \tag{2.4}$$

Suppose now that the transformation matrix $\mathbf{A}$ fulfills the condition

$$\mathbf{A}\mathbf{k}_n\mathbf{A}^{\mathrm{T}} = \alpha^2\mathbf{I}_n, \quad \text{or, equivalently,} \quad \mathbf{A}^{\mathrm{T}}\mathbf{A} = \alpha^2\mathbf{k}_n^{-1}, \tag{2.5}$$

where $\mathbf{I}_n$ is an $(n \times n)$ unit matrix and α is an arbitrarily assigned constant. By taking the determinant of (2.5), it can be readily seen that,

$$(\det\mathbf{A})^2 = \alpha^{2n}\det(\mathbf{k}_n^{-1}), \quad \text{or equivalently,} \quad \det\mathbf{A} = \alpha^n/\det\mathbf{k}_n^{\frac{1}{2}}. \tag{2.6}$$

Thus (2.4) can be recast as

$$\alpha^{2-n}\kappa_n\nabla'^2 G + \delta(\mathbf{x}' - \mathbf{x}'_0) = 0, \tag{2.7}$$

entirely similar to that of an isotropic solid, where

$$\kappa_n \equiv \det\mathbf{k}_n^{1/2}, \quad \nabla'^2 \equiv \frac{\partial^2}{\partial x_1'^2} + \frac{\partial^2}{\partial x_2'^2} + \cdots + \frac{\partial^2}{\partial x_n'^2}. \tag{2.8}$$

At the present stage, there is no need to discuss the detailed solution of $\mathbf{A}$ that satisfies (2.5). We will show how to obtain the explicit forms of $\mathbf{A}$ in Appendix A, which will be of crucial importance in subsequent developments. The above procedure suggests that the Green's function for a general anisotropic conductivity $\mathbf{k}_n$ can be formulated similar to an equivalent isotropic conductivity of value $\alpha^{2-n}\kappa_n$. Mathematically, this implies that a general elliptic second-order partial differential equation can be transformed into a canonical form, the Laplace equation. Since the Green's function for the Laplacian operator can be readily found in the literature, one can construct the anisotropic Green's function G in the physical domain (in $\mathbf{x}$-coordinate) by back transformation from the $\mathbf{x}'$ space.

Here we consider the Green's functions for two-, three- and higher-dimensional spaces in turn. For convenience, we record the isotropic Green's functions in Appendix B. We observe that, for a two-dimensional plane, by letting $n = 2$ in (2.7), the equivalent isotropic conductivity is simply κ_2, irrelevant to the value of α. Noting that

$$\begin{aligned}|\mathbf{x}' - \mathbf{x}'_0| &= [(\mathbf{x}' - \mathbf{x}'_0)^{\mathrm{T}}(\mathbf{x}' - \mathbf{x}'_0)]^{1/2} = |\mathbf{A}(\mathbf{x} - \mathbf{x}_0)| = [(\mathbf{x} - \mathbf{x}_0)^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{A}(\mathbf{x} - \mathbf{x}_0)]^{1/2} \\ &= \alpha[(\mathbf{x} - \mathbf{x}_0)^{\mathrm{T}}\mathbf{k}_n^{-1}(\mathbf{x} - \mathbf{x}_0)]^{1/2} \equiv \alpha R,\end{aligned} \tag{2.9}$$

the isotropic Green's function G (B.1) can then be transformed back into the $\mathbf{x}$-coordinates as

$$G_{2\mathrm{D}}^{\infty}(\mathbf{x}; \mathbf{x}_0) = -\frac{\log R}{2\pi\kappa_2}. \tag{2.10}$$

Without loss of any generality, an additive constant in (2.10) is omitted.

For an n-dimensional space with $n \geqslant 3$, in reference to (B.6), we have

$$G_{n\mathrm{D}}^{\infty}(\mathbf{x}; \mathbf{x}_0) = \frac{R^{2-n}}{(n-2)\omega_n\kappa_n}, \quad \omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \quad \text{for } n \geqslant 3. \tag{2.11}$$

Here ω_n denotes the surface area of the unit sphere in an n-dimensional space $\mathbb{R}^n$ and $\Gamma(\cdot)$ is the factorial function. Note that the solutions for two- and higher-dimensional spaces are mathematically different. The present formulation offers a simple and unified approach to construct the Green's functions in conduction for all dimensional spaces. We mention that the above results agree with the known solutions for an unbounded plane $n = 2$ and space $n = 3$ (Chang, 1977, Eqs. (4.6) and (4.7)). To our knowledge, the solutions for $n > 3$, (2.11), are new.

3. Green's function for a half-space

In this section, we examine the anisotropic Green's function in conduction for an n-dimensional half-space. We suppose that the half-space occupies the region Ω: $0 \leqslant x_1 < \infty$. The field equation under steady-state condition subject to a point heat source at $\mathbf{x}_0$ has the form (2.1). Along the boundary of the domain $\partial\Omega$: $F(\mathbf{x})$: $x_1 = 0$, three different kinds of homogeneous boundary conditions are considered. For the first-kind boundary condition the temperature is set equal to zero along the boundary; for the second-kind the normal component of heat flux is taken as zero; for the third kind a certain combination of the temperature and the normal component of heat flux is set to zero. These boundary conditions are expressed, respectively, as (Özişik, 1993)

$$G^{\mathrm{I}}|_{x_1=0} = 0, \quad \text{or} \quad \mathbf{k}_n\nabla G^{\mathrm{II}} \cdot \mathbf{n}|_{x_1=0} = 0, \quad \text{or} \quad \mathbf{k}_n\nabla G^{\mathrm{III}} \cdot \mathbf{n} + hG^{\mathrm{III}}|_{x_1=0} = 0. \tag{3.1}$$

Here h is the known convective heat transfer coefficient and $\mathbf{n}$ is the unit outward normal to the boundary $\partial\Omega$. For distinction, we designate the corresponding Green's functions associated with the three kinds of boundary conditions by superscripts I, II, III.

To proceed, we again make use of the affine transformation (2.2). The field equation after the transformation remains the same as that for an unbounded space and given by (2.7). For the boundary conditions, we need to discuss more on the transformation matrix $\mathbf{A}$ defined in (2.5). As shown in Appendix A, the transformation matrix $\mathbf{A}$ can be found from the Cholesky factorization, which takes the form of a lower triangular matrix. Particularly, for all dimensionality n, it is seen that the component of A_{11} has the unique value $\alpha/\sqrt{k_{11}}$. To examine the boundary term under the transformation, we first note that the new boundary of the transformed domain Ω', designated as $\partial\Omega'$, is simply $x_1' = 0$. Next, we need to express the three kinds of boundary conditions in the $\mathbf{x}'$-space. It is easy to see that the first-kind boundary condition remains unchanged under the transformation. The key step in the formulation is the transformation of the second-kind boundary condition, which will also be used in deriving the third-type boundary condition. We derive $(3.1)_2$ in the following steps

$$\begin{aligned}\mathbf{k}_n\nabla G\cdot\mathbf{n} &= \mathbf{k}_n\mathbf{A}^{\mathrm{T}}\nabla' G\cdot\left(-\frac{\nabla F}{|\nabla F|}\right) = -\mathbf{k}_n\mathbf{A}^{\mathrm{T}}\nabla' G\cdot\frac{\mathbf{A}^{\mathrm{T}}\nabla' F}{|\nabla F|} = -\mathbf{A}\mathbf{k}_n\mathbf{A}^{\mathrm{T}}\nabla' G\cdot\frac{\nabla' F}{|\nabla F|}\\ &= \alpha^2\nabla' G\cdot\left(-\frac{\nabla' F}{|\nabla' F|}\right)\frac{|\nabla' F|}{|\nabla F|} = \alpha^2\frac{|\nabla' F|}{|\nabla F|}(\nabla' G\cdot\mathbf{n}').\end{aligned} \tag{3.2}$$

Since

$$F(\mathbf{x}) = F(\mathbf{A}^{-1}\mathbf{x}') = x_1 = \alpha^{-1}k_{11}^{\frac{1}{2}}x_1' = 0, \tag{3.3}$$

there follows

$$\frac{|\nabla' F|}{|\nabla F|} = \alpha^{-1}k_{11}^{\frac{1}{2}}. \tag{3.4}$$

With these developments (3.2)–(3.4), the three different kinds of boundary conditions (3.1) are then transformed into

$$G^{\mathrm{I}}|_{x_1'=0} = 0, \quad \nabla' G^{\mathrm{II}}\cdot\mathbf{n}'|_{x_1'=0} = 0, \quad \nabla' G^{\mathrm{III}}\cdot\mathbf{n}' + \tilde{p}G^{\mathrm{III}}|_{x_1'=0} = 0, \tag{3.5}$$

where $\mathbf{n}'$ is the unit normal to $\partial\Omega'$ and

$$\tilde{p} \equiv h\alpha^{-1}k_{11}^{-\frac{1}{2}}. \tag{3.6}$$

Thus, (2.7) along with the boundary conditions (3.5) in the $\mathbf{x}'$-space become exactly the same as those for the corresponding half-space Green's functions with an isotropic conductivity. Since the latter solutions can be found in the literature, see Eqs. (B.2), (B.3) and (B.7)–(B.9) of Appendix B, one can thus obtain the anisotropic Green's function in the physical domain $\mathbf{x}$ without resolving the field equations.

Here we derive the half-space Green's functions for two-, three- and higher dimensions. For a two-dimensional plane, the half-plane Green's functions related to the first- and the second-kind of boundary conditions are obtained as

$$G_{2\mathrm{D}}^{\mathrm{I,II}}(\mathbf{x};\mathbf{x}_0) = -\frac{\log R}{2\pi\kappa_2} \pm \frac{\log R_i}{2\pi\kappa_2}, \tag{3.7}$$

where R has been defined in (2.9) and R_i is defined as

$$\alpha R_i \equiv |\mathbf{x}' - \mathbf{x}_0^{i\prime}| = |\mathbf{A}\mathbf{x} - \mathbf{R}_2\mathbf{A}\mathbf{x}_0|. \tag{3.8}$$

Here $\mathbf{R}_2$ is a (2×2) reflectional matrix defined in Appendix B. The Green's function corresponding to the third-kind boundary condition requires few more steps of derivation. In the transformed space, the Green's function in an equivalent isotropic medium was recorded in (B.3) and (B.4). To transform (B.3) back into the physical space $\mathbf{x}$, we see that the first two terms in (B.3) were just R and R_i given in (2.9) and (3.8). The third term is, however, more complicated and can be formulated as follows. We first change the dummy index η in the $\mathbf{x}'$-space to a dummy index ξ in the $\mathbf{x}$-space by noting $\eta = A_{11}\xi$ and $\mathbf{x}_0' = \mathbf{A}\mathbf{x}_0$. Thus we have

$$\tilde{h}(\eta) = -2\tilde{p}\exp(A_{11}\tilde{p}(x_1^0+\xi)), \quad \mathbf{x}_\eta'^i = \begin{pmatrix} A_{11}\xi \\ A_{21}x_1^0 + A_{22}x_2^0 \end{pmatrix}, \tag{3.9}$$

and

$$\begin{aligned} \int_{-\infty}^{x_1'^0} \tilde{h}(\eta)\frac{\log|\mathbf{x}'-\mathbf{x}_\eta'^i|}{2\pi k}\mathrm{d}\eta &= \int_{-\infty}^{-x_1^0} \tilde{h}(\eta)\frac{\log|\mathbf{x}'-\mathbf{x}_\eta'^i|}{2\pi k}(A_{11}\mathrm{d}\xi) \\ &= \int_{-\infty}^{-x_1^0} -2p\exp(p(x_1^0+\xi))\frac{\log|\mathbf{x}'-\mathbf{x}_\eta'^i|}{2\pi k}\mathrm{d}\xi \\ &= \int_0^{\infty} -2p\exp(-p\zeta)\frac{\log|\mathbf{x}'-\mathbf{x}_\eta'^i|}{2\pi k}\mathrm{d}\zeta, \end{aligned} \tag{3.10}$$

where $p \equiv hk_{11}^{-1}$ and $\zeta = -x_1^0 - \xi$. Further we can write

$$|\mathbf{x}'-\mathbf{x}_\eta'^i| = \left|\mathbf{x}' - \begin{pmatrix} A_{11}(-x_1^0-\zeta) \\ A_{21}x_1^0 + A_{22}x_2^0 \end{pmatrix}\right| = |\mathbf{Ax} - \mathbf{R}_2\mathbf{Ax}_0 + \zeta\mathbf{Ae}_1| \equiv \alpha R_\zeta, \tag{3.11}$$

with $\mathbf{e}_1 = (1,0)^{\mathrm{T}}$. Thus the Green's function for a half-space with the third-kind boundary condition can be expressed as

$$G_{2\mathrm{D}}^{\mathrm{III}}(\mathbf{x};\mathbf{x}_0) = -\frac{\log R}{2\pi\kappa_2} - \frac{\log R_i}{2\pi\kappa_2} - \int_0^\infty h(\zeta)\frac{\log R_\zeta}{2\pi\kappa_2}\mathrm{d}\zeta, \tag{3.12}$$

where

$$h(\zeta) = -2p\exp(-p\zeta). \tag{3.13}$$

We mention that the quantities R_i and R_ζ, after some manipulations, can be simplified as

$$R_i^2 = R^2 + \frac{4x_1x_1^0}{k_{11}}, \quad R_\zeta^2 = R^2 + \frac{(2x_1+\zeta)(2x_1^0+\zeta)}{k_{11}}. \tag{3.14}$$

Eqs. (3.7) and (3.12) recover the known Green's function for an anisotropic half-plane (Chang, 1977, Eqs. (5.9) and (5.11)).

For an n-dimensional space with $n \geqslant 3$, the half-space Green's functions for conduction have the forms

$$G_{n\mathrm{D}}^{\mathrm{I,II}}(\mathbf{x};\mathbf{x}_0) = \frac{R^{2-n}}{(n-2)\omega_n\kappa_n} \mp \frac{R_i^{2-n}}{(n-2)\omega_n\kappa_n}, \tag{3.15}$$

and

$$G_{n\mathrm{D}}^{\mathrm{III}}(\mathbf{x};\mathbf{x}_0) = \frac{R^{2-n}}{(n-2)\omega_n\kappa_n} + \frac{R_i^{2-n}}{(n-2)\omega_n\kappa_n} + \int_0^\infty h(\zeta)\frac{R_\zeta^{2-n}}{(n-2)\omega_n\kappa_n}\mathrm{d}\zeta, \tag{3.16}$$

where now

$$\alpha R_i = |\mathbf{Ax} - \mathbf{AR}_n\mathbf{x}_0|, \quad \alpha R_\zeta = |\mathbf{Ax} - \mathbf{R}_n\mathbf{Ax}_0 + \zeta\mathbf{Ae}_1|, \tag{3.17}$$

and $\mathbf{e}_1 = (1, 0, \ldots, 0)_{n\times 1}^{\mathrm{T}}$. For $n = 3$, Eqs. (3.15) and (3.16) also agree with the known Green's functions for a half-space (Chang, 1977, Eqs. (5.4) and (5.6)). But here we remark that Chang's derivations (1977) are rather tedious. To our knowledge, the Green's functions for $n > 3$ are new in the literature.

4. Green's function for bimaterials

This section is concerned with Green's functions for a bimaterial. We consider that the bimaterial is composed of two anisotropic half-spaces bonded along $x_1 = 0$ in an n-dimensional space. The material A occupies the region Ω_A: $x_1 > 0$ and material B the remaining half-space Ω_B: $x_1 < 0$. We suppose that a point heat source is prescribed at a point $\mathbf{x}_0$ inside the region A. The field equilibrium equations under steady-state conditions are

$$\begin{aligned} &k_{ij}^A \frac{\partial^2 G^A}{\partial x_i \partial x_j} + \delta(\mathbf{x} - \mathbf{x}_0) = 0, \quad \text{for } \mathbf{x} \in \Omega_A, \\ &k_{ij}^B \frac{\partial^2 G^B}{\partial x_i \partial x_j} = 0, \quad \text{for } \mathbf{x} \in \Omega_B. \end{aligned} \tag{4.1}$$

We assume that the interface, J: $x_1 = 0$, is perfectly bonded, which means that the temperature and the normal component of heat flux are continuous across the interface J

$$\begin{aligned} &G^A|_J = G^B|_J, \\ &\mathbf{k}_n^A \nabla G^A \cdot \mathbf{n}|_J = \mathbf{k}_n^B \nabla G^B \cdot \mathbf{n}|_J. \end{aligned} \tag{4.2}$$

For convenience, we can rewrite the continuity condition $(4.2)_2$ as

$$k_{1j}^A \frac{\partial G^A}{\partial x_j}\bigg|_{x_1=0} = k_{1j}^B \frac{\partial G^B}{\partial x_j}\bigg|_{x_1=0}. \tag{4.3}$$

To proceed, let us introduce affine coordinate transformations

$$\mathbf{x}' = \mathbf{A}_A \mathbf{x}, \quad \text{for } \mathbf{x} \in \Omega_A, \qquad \mathbf{x}'' = \mathbf{A}_B \mathbf{x}, \quad \text{for } \mathbf{x} \in \Omega_B, \tag{4.4}$$

for the regions A and B, in which

$$\mathbf{A}_A^{\mathrm{T}} \mathbf{A}_A = \alpha_A^2 (\mathbf{k}_n^A)^{-1}, \quad \mathbf{A}_B^{\mathrm{T}} \mathbf{A}_B = \alpha_B^2 (\mathbf{k}_n^B)^{-1}. \tag{4.5}$$

Following the routes outlined in (2.3) and (3.2), the field equations are changed to

$$\begin{aligned} &\alpha_A^{2-n} \kappa_n^A \nabla'^2 G^A + \delta(\mathbf{x}' - \mathbf{x}'_0) = 0, \quad \text{for } \mathbf{x}' \in \Omega'_A, \\ &\alpha_B^{2-n} \kappa_n^B \nabla''^2 G^B = 0, \quad \text{for } \mathbf{x}'' \in \Omega''_B \end{aligned} \tag{4.6}$$

and the interface continuity conditions become

$$G^A|_{x'_1=0} = G^B|_{x''_1=0}, \quad \alpha_A \sqrt{k_{11}^A} \frac{\partial G^A}{\partial x'_1}\bigg|_{x'_1=0} = \alpha_B \sqrt{k_{11}^B} \frac{\partial G^B}{\partial x''_1}\bigg|_{x''_1=0}. \tag{4.7}$$

Note that in deriving $(4.7)_2$, we have made use of the relations (3.2) and (3.4). Also, in line with (2.8), we have defined the following short notations

$$\kappa_n^I = \det(\mathbf{k}_n^I)^{\frac{1}{2}}, \;\; I = A, B, \quad \nabla''^2 = \frac{\partial^2}{\partial x_1''^2} + \frac{\partial^2}{\partial x_2''^2} + \cdots + \frac{\partial^2}{\partial x_n''^2}. \tag{4.8}$$

For later convenience, we will also define

$$m_n^{ij} \equiv (-1)^{i+j} \det \mathbf{k}_n^{ij}, \tag{4.9}$$

where $\mathbf{k}_n^{ij}$ is the $(n-1) \times (n-1)$ submatrix of the n-dimensional conductivity matrix $\mathbf{k}_n$ by deleting the row and the column containing the element $(\mathbf{k}_n)_{ij}$.

4.1. Two-dimensional plane

Here we first consider the two-dimensional plane, i.e., $n = 2$. In this case, the transformation matrix $\mathbf{A}$ can be found explicitly as (see Appendix A)

$$\mathbf{A} = \alpha \begin{pmatrix} \frac{1}{\kappa_1} & 0 \\ \frac{m_2^{21}}{\kappa_1\kappa_2} & \frac{m_2^{22}}{\kappa_1\kappa_2} \end{pmatrix}, \tag{4.10}$$

where, in accordance with the definitions in (4.9) and (4.8), $\kappa_1 = \sqrt{k_{11}}$, $\kappa_2 = \sqrt{k_{11}k_{22} - k_{12}^2}$, $m_2^{22} = k_{11}$ and $m_2^{21} = -k_{12}$. Here we have omitted the material designation A and B. To proceed, first upon the transformation (4.4) it is evident that the interface $x_1 = 0$ is mapped, respectively, onto $x_1' = 0$ and $x_1'' = 0$. Next, the points $(0, x_2)$ along the interface J in the $\mathbf{x}$-space are carried onto points $(0, x_2')$ in the $\mathbf{x}'$-space and to points $(0, x_2'')$ in the $\mathbf{x}''$-space, via the transformation (4.4), in which

$$x_2' = \alpha_A(\kappa_2^A)^{-1}\kappa_1^A x_2, \quad x_2'' = \alpha_B(\kappa_2^B)^{-1}\kappa_1^B x_2. \tag{4.11}$$

We need to enforce the condition $x_2' = x_2''$ so that the interfacial points remain intact under the transformation. A simple choice that fulfills the condition is

$$\frac{\alpha_B}{\alpha_A} = \frac{\frac{\kappa_1^A}{\kappa_2^A}}{\frac{\kappa_1^B}{\kappa_2^B}}. \tag{4.12}$$

Upon substitution (4.12) back into (4.6) and (4.7), the governing set for the transformed fields becomes

$$\begin{aligned} &\kappa_2^A \nabla'^2 G^A + \delta(\mathbf{x}' - \mathbf{x}_0') = 0, \quad \text{for } \mathbf{x}' \in \Omega_A', \\ &\kappa_2^B \nabla''^2 G^B = 0, \quad \text{for } \mathbf{x}'' \in \Omega_A'' \end{aligned} \tag{4.13}$$

and the interface conditions reduce to

$$G^A|_{x_1'=0} = G^B|_{x_1''=0}, \quad \kappa_2^A \frac{\partial G^A}{\partial x_1'}\bigg|_{x_1'=0} = \kappa_2^B \frac{\partial G^B}{\partial x_1''}\bigg|_{x_1''=0}. \tag{4.14}$$

Remarkably, the system becomes now exactly identical to that of the Green's function for a bimaterial with isotropic conductivities, κ_2^A and κ_2^B, in each half-plane. For the latter solutions, see Eqs. (C.1) and (C.2). Thus by back transformation, we find the Green's function for an anisotropic bimaterial in the physical space $\mathbf{x}$ as

$$G_{2D}^A(\mathbf{x}; \mathbf{x}_0) = -\frac{\log R_A}{2\pi\kappa_2^A} - \frac{\log R_{A,i}}{2\pi\kappa_2^A}\frac{\kappa_2^A - \kappa_2^B}{\kappa_2^A + \kappa_2^B}, \tag{4.15}$$

$$G_{2D}^B(\mathbf{x}; \mathbf{x}_0) = -\frac{\log R_B}{\pi(\kappa_2^A + \kappa_2^B)}, \tag{4.16}$$

where

$$R_A = \mid \mathbf{A}_A(\mathbf{x} - \mathbf{x}_0) \mid, \quad R_{A,i} = \mid \mathbf{A}_A\mathbf{x} - \mathbf{R}_2\mathbf{A}_A\mathbf{x}_0 \mid, \quad R_B = \mid \mathbf{A}_B\mathbf{x} - \mathbf{A}_A\mathbf{x}_0 \mid. \tag{4.17}$$

We have verified that the above expressions for the Green's functions for a two-dimensional bimaterial exactly agree with a mathematically equivalent problem of Green's function of anti-plane elasticity (Ting, 1996), in which the Green's function was derived using the Stroh formalism. Also, our results are verified with the Green's functions in conduction of a two-dimensional bimaterial by Berger et al. (2000) in which the solutions were derived from the Fourier integral transform. The present formulation could be viewed as an alternative approach for the considered two-dimensional case and yet it is comparatively simple and straightforward.

4.2. Three-dimensional space

Next we consider that the bimaterial is constituted by two three-dimensional half-spaces. Again, the interface is characterized by $x_1 = 0$. In this case, the transformation matrix $\mathbf{A}$ has the explicit form

$$\mathbf{A} = \alpha \begin{pmatrix} \frac{1}{\kappa_1} & 0 & 0 \\ \frac{m_2^{21}}{\kappa_1\kappa_2} & \frac{m_2^{22}}{\kappa_1\kappa_2} & 0 \\ \frac{m_3^{31}}{\kappa_2\kappa_3} & \frac{m_3^{32}}{\kappa_2\kappa_3} & \frac{m_3^{33}}{\kappa_2\kappa_3} \end{pmatrix}. \tag{4.18}$$

Upon the transformations (4.4), the interface $x_1 = 0$ is again mapped, respectively, onto $x_1' = 0$ and $x_1'' = 0$. Next, for the problem to be physically realistic, we need to ensure that the points $(0, x_2, x_3)$ along the interfacial plane J in the $\mathbf{x}$-space are carried onto the same position $(0, x_2', x_3')$ in the $\mathbf{x}'$-space and $(0, x_2'', x_3'')$ in the $\mathbf{x}''$-space, namely $x_2' = x_2''$ and $x_3' = x_3''$. For this to be true, it is necessary that the components of A_{22}, A_{33} and A_{32} be identical for the two materials A and B (4.4). This implies that the conductivities of the two constituent phases and the parameters α_A and α_B must fulfill the algebraic constraints

$$\alpha_A \frac{m_2^{22A}}{\kappa_1^A\kappa_2^A} = \alpha_B \frac{m_2^{22B}}{\kappa_1^B\kappa_2^B}, \quad \alpha_A \frac{m_3^{32A}}{\kappa_2^A\kappa_3^A} = \alpha_B \frac{m_3^{32B}}{\kappa_2^B\kappa_3^B}, \quad \alpha_A \frac{m_3^{33A}}{\kappa_2^A\kappa_3^A} = \alpha_B \frac{m_3^{33B}}{\kappa_2^B\kappa_3^B}, \tag{4.19}$$

or, equivalently,

$$\frac{\frac{m_2^{22A}}{\kappa_1^A\kappa_2^A}}{\frac{m_2^{22B}}{\kappa_1^B\kappa_2^B}} = \frac{\frac{m_3^{32A}}{\kappa_2^A\kappa_3^A}}{\frac{m_3^{32B}}{\kappa_2^B\kappa_3^B}} = \frac{\frac{m_3^{33A}}{\kappa_2^A\kappa_3^A}}{\frac{m_3^{33B}}{\kappa_2^B\kappa_3^B}} = \frac{\alpha_B}{\alpha_A}. \tag{4.20}$$

We note that the first equality of (4.20) is exactly the proportional ratio (4.12) that we selected for a two-dimensional plane. In view of the constraint condition, it is seen that if the phase conductivities fulfill the first two equalities in (4.20), then the ratio of α_B/α_A is fixed accordingly. This means that only one of the two scalars, α_A and α_B, can be arbitrarily assigned. In particular, if one chooses $\alpha_A = \kappa_3^A/\kappa_2^A$, then it follows that $\alpha_B = \kappa_3^B/\kappa_2^B$. Interestingly, with these choices, the governing systems (4.6) and (4.7) are then reduced to (4.13) and (4.14), in a form entirely identical to that of a two-dimensional bimaterial. Of course here the coordinates $\mathbf{x}'$ and $\mathbf{x}''$ are for three-dimensional spaces. Note that in deriving (4.7), we have made use of the connections (4.20). Also, it should be mentioned that for a given positive-definite conductivity matrix for material A, we have found numerically that there exist numerous (positive-definite) conductivity matrices for material B that fulfill the constraint (4.20). In summary, for the Green's function of a three-dimensional anisotropic bimaterial, if the conductivities of the two phases fulfill the first two equalities in (4.20), then one can adjust the ratio of α_B/α_A to satisfy (4.20). Under such circumstances, the affine transformation (4.4) with (4.5) will transform the physical problem into that for a three-dimensional *isotropic bimaterial* with phase isotropic conductivities given by κ_2^A and κ_2^B. As the Green's function for an isotropic bimaterial are recorded in (C.3) and (C.4) [1] by mapping back onto the physical domain, we can obtain the Green's function for a three-dimensional anisotropic bimaterial as

[1] We are not aware of the existence of Green's function in conduction for three- and n-dimensional isotropic bimaterials. We have derived the Green's functions for a three-dimensional isotropic bimaterial using an integral transform method. The process, though straightforward, is rather tedious. For brevity, only final solutions are recorded in Appendix C.

$$G^A_{3D}(\mathbf{x};\mathbf{x}_0)=\frac{1}{4\pi\kappa^A_2 R_A}+\frac{1}{4\pi\kappa^A_2 R_{A,i}}\frac{\kappa^A_2-\kappa^B_2}{\kappa^A_2+\kappa^B_2}, \tag{4.21}$$

$$G^B_{3D}(\mathbf{x},\mathbf{x}_0)=\frac{1}{4\pi\kappa^B_2 R_B}\frac{2\kappa^B_2}{\kappa^A_2+\kappa^B_2}, \tag{4.22}$$

where R_A and R_B have the same expressions in (4.17) and

$$R_{A,i}=|\ \mathbf{A}_A\mathbf{x}-\mathbf{R}_3\mathbf{A}_A\mathbf{x}_0\ |. \tag{4.23}$$

4.3. Higher-dimensional space

The foregoing concept can be extended to a higher-dimensional bimaterial. For an n-dimensional space, the transformation matrix $\mathbf{A}$ can be proven as

$$\mathbf{A}=\alpha\begin{pmatrix} \frac{1}{\kappa_1} & 0 & 0 & \cdots & 0 \\ \frac{m_2^{21}}{\kappa_1\kappa_2} & \frac{m_2^{22}}{\kappa_1\kappa_2} & 0 & \cdots & 0 \\ \frac{m_3^{31}}{\kappa_2\kappa_3} & \frac{m_3^{32}}{\kappa_2\kappa_3} & \frac{m_3^{33}}{\kappa_2\kappa_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{m_n^{n1}}{\kappa_{n-1}\kappa_n} & \frac{m_n^{n2}}{\kappa_{n-1}\kappa_n} & \frac{m_n^{n3}}{\kappa_{n-1}\kappa_n} & \cdots & \frac{m_n^{nn}}{\kappa_{n-1}\kappa_n} \end{pmatrix}. \tag{4.24}$$

To ensure that the interfacial points $x_1=0$ will perfectly match after two different affine transformations (4.4), we need to demand that all the elements of $\mathbf{A}$ except the first column for A_A and A_B be identical. That is,

$$\frac{\frac{m_2^{22A}}{\kappa_1^A\kappa_2^A}}{\frac{m_2^{22B}}{\kappa_1^B\kappa_2^B}}=\frac{\frac{m_3^{32A}}{\kappa_2^A\kappa_3^A}}{\frac{m_3^{32B}}{\kappa_2^B\kappa_3^B}}=\cdots=\frac{\frac{m_n^{n2A}}{\kappa_{n-1}^A\kappa_n^A}}{\frac{m_n^{n2B}}{\kappa_{n-1}^B\kappa_n^B}}=\frac{\frac{m_3^{33A}}{\kappa_2^A\kappa_3^A}}{\frac{m_2^{33B}}{\kappa_2^B\kappa_3^B}}=\cdots=\frac{\frac{m_n^{n3A}}{\kappa_{n-1}^A\kappa_n^A}}{\frac{m_n^{n3B}}{\kappa_{n-1}^B\kappa_n^B}}=\cdots=\frac{\frac{m_n^{nnA}}{\kappa_{n-1}^A\kappa_n^A}}{\frac{m_n^{nnB}}{\kappa_{n-1}^B\kappa_n^B}}=\frac{\alpha_B}{\alpha_A}. \tag{4.25}$$

These constitute a total of $n(n-1)/2$ conditions among the phase conductivities $\mathbf{k}^A_n$, $\mathbf{k}^B_n$. Similar to the reasoning for the three-dimensional case, the value of α_A can be arbitrarily assigned. Without loss of any generality, one can choose $\alpha_A=(\kappa^A_n/\kappa^A_2)^{\frac{1}{n-2}}$. Under such circumstances, as in two- and three-dimensional bimaterials, the governing system for the Green's functions of an n-dimensional anisotropic bimaterial is exactly the same as that for an n-dimensional isotropic bimaterial with equivalent conductivities κ^A_2 and κ^B_2. As the latter solutions have been found in Appendix C, we thus can write the anisotropic Green's functions in the physical space $\mathbf{x}$ as

$$G^A_{nD}(\mathbf{x};\mathbf{x}_0)=\frac{R_A^{2-n}}{(n-2)\omega_n\kappa^A_2}+\frac{R_{A,i}^{2-n}}{(n-2)\omega_n\kappa^A_2}\frac{\kappa^A_2-\kappa^B_2}{\kappa^A_2+\kappa^B_2}, \tag{4.26}$$

$$G^B_{nD}(\mathbf{x};\mathbf{x}_0)=\frac{R_B^{2-n}}{(n-2)\omega_n\kappa^B_2}\frac{2\kappa^B_2}{\kappa^A_2+\kappa^B_2}, \tag{4.27}$$

where again R_A, $R_{A,i}$ and R_B have been defined in (4.17), except that the matrix $\mathbf{R}_2$ now should be replaced by $\mathbf{R}_n$. The Green's functions for three- and higher-dimensional bimaterials, though only applicable to restricted systems, are not known in the literature (T.C.T. Ting, private communication).

5. Green's functions for a layered space

In this section we examine Green's functions in conduction for an n-dimensional, anisotropic, layered medium. The medium consists of m different constituent layers which are all perfectly bonded together. The pth layer, with the conductivity tensor denoted by $\mathbf{k}_n^{(p)}$, occupies the region Ω_p: $h_p \leqslant x_1 \leqslant h_{p-1}$, $p = 1, 2, \ldots, m$, for a referenced Cartesian coordinate. Here we have set $h_m \to -\infty$ and $h_0 \to \infty$, so that the layered medium occupies the full space. Suppose that a point heat source is applied at a point x_0 inside the qth layer Ω_q. The field equilibrium equations under steady-state conditions are

$$k_{ij}^{(p)} \frac{\partial^2 G^{(p)}}{\partial x_i \partial x_j} + \delta_{pq}\delta(\mathbf{x} - \mathbf{x}_0) = 0, \quad \text{for } \mathbf{x} \in \Omega_p,\ \mathbf{x}_0 \in \Omega_q,\ p = 1, \ldots, m, \tag{5.1}$$

where δ_{pq} is the Kronecker delta. The interface J_p: $x_1 = h_p$, $p = 1, 2, \ldots, m-1$, between the two adjacent regions, Ω_p and Ω_{p+1}, is perfectly bonded so that one has

$$\begin{aligned} &G^{(p)}|_{J_p} = G^{(p+1)}|_{J_p}, \\ &\mathbf{k}_n^{(p)} \nabla G^{(p)} \cdot \mathbf{n}|_{J_p} = \mathbf{k}_n^{(p+1)} \nabla G^{(p+1)} \cdot \mathbf{n}|_{J_p}. \end{aligned} \tag{5.2}$$

To proceed, we make use of the affine transformation. But, in contrast to (4.4), we now introduce

$$\mathbf{x}'^{(p)} = \mathbf{A}^{(p)}\mathbf{x} + \mathbf{a}^{(p)}, \ \text{ for } \mathbf{x} \in \Omega_p, \quad \mathbf{A}^{(p)\mathrm{T}}\mathbf{A}^{(p)} = \alpha_p^2 (\mathbf{k}_n^{(p)})^{-1}, \tag{5.3}$$

for the region Ω_p with an additional shifting $(n \times 1)$ vector $\mathbf{a}^{(p)}$. Here $\mathbf{x}'^{(p)}$, $p = 1, \ldots, m$, is the transformed coordinate for the region Ω_p [2] and $\mathbf{a}^{(p)}$ is an n-dimensional column vector to be determined. This implies that each region Ω_p has its own transformation matrix $\mathbf{A}^{(p)}$ together with a shifting vector $\mathbf{a}^{(p)}$. For convenience we shall denote the transformed region for the region Ω_p as Ω'_p. We shall see that a layered space will remain as a layered space after the transformation (5.3), but with different spacings. As before, the idea is to transform the physical problem into a transformed domain with isotropic constituents. To resolve the field equations for the transformed configuration, it is necessary that the interfacial points J_p between any two adjacent regions Ω_p and Ω_{p+1} must meet perfectly upon the transformation (5.3). That is we need to demand the equality

$$\mathbf{A}^{(p)} \begin{pmatrix} h_p \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \mathbf{a}^{(p)} = \mathbf{A}^{(p+1)} \begin{pmatrix} h_p \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \mathbf{a}^{(p+1)}, \quad \text{for } p = 1, \ldots, m-1. \tag{5.4}$$

Since $A_{ij}^{(p)}$ is a lower triangular matrix (4.24) and also due to the fact that $x_2, \ldots, x_n$ could be arbitrary, the conditions (5.4) are satisfied if and only if the phase properties satisfy the algebraic constraints

$$A_{ij}^{(p)} = A_{ij}^{(p+1)}, \quad 2 \leqslant i \leqslant n;\ \ 2 \leqslant j \leqslant i, \tag{5.5}$$

and that the shifting vectors are selected as

$$a_i^{(p+1)} = a_i^{(p)} + (A_{i1}^{(p)} - A_{i1}^{(p+1)})h_p, \quad i = 1, \ldots, n. \tag{5.6}$$

[2] Note that in Section 4, the transformed coordinates for region A and region B for a bimaterial are, respectively, designated by $\mathbf{x}'$ and $\mathbf{x}''$.

Recalling (4.24), (5.5) can be rephrased as

$$\alpha_1 \frac{m_i^{ij(1)}}{\kappa_{i-1}^{(1)}\kappa_i^{(1)}} = \alpha_2 \frac{m_i^{ij(2)}}{\kappa_{i-1}^{(2)}\kappa_i^{(2)}} = \cdots = \alpha_m \frac{m_i^{ij(m)}}{\kappa_{i-1}^{(m)}\kappa_i^{(m)}}, \quad 2 \leqslant i \leqslant n;\ 2 \leqslant j \leqslant i. \tag{5.7}$$

The above condition implies that the phase conductivities for any two constituent regions need to fulfill the constraints (4.25). Further, Eq. (5.6) provides a recurrent formula for the determination of $\mathbf{a}^{(p)}$, $p = 2,\ldots,m$. Without loss of any generality, one can set $\mathbf{a}^{(1)} = \mathbf{0}$ in (5.6) so that the components of $\mathbf{a}^{(p)}$ can be explicitly written as:

$$\begin{aligned} a_1^{(p)} &= \sum_{d=1}^{p-1} \left(\frac{\alpha_d}{\kappa_1^{(d)}} - \frac{\alpha_{d+1}}{\kappa_1^{(d+1)}} \right) h_d, \quad p = 2,\ldots,m, \\ a_i^{(p)} &= \sum_{d=1}^{p-1} \left(\alpha_d \frac{m_i^{i1(d)}}{\kappa_{i-1}^{(d)}\kappa_i^{(d)}} - \alpha_{d+1} \frac{m_i^{i1(d+1)}}{\kappa_{i-1}^{(d+1)}\kappa_i^{(d+1)}} \right) h_d, \quad 2 \leqslant i \leqslant n,\ p = 2,\ldots,m. \end{aligned} \tag{5.8}$$

Under the conditions (5.4), the transformed interfaces J'_p remain as planes (hyper-planes), described as

$$J'_p : x_1'^{(p)} = x_1'^{(p+1)} = \frac{\alpha_p}{\kappa_1^{(p)}} h_p + a_1^{(p)}, \quad p = 1,\ldots,m-1. \tag{5.9}$$

Here the selection of the constants α_p will be discussed later. Back to (5.1), upon the transformation (5.3), the field equation is now changed to

$$\kappa_2^{(p)} \nabla'^2 G^p + \delta_{pq}\delta(\mathbf{x}'^{(p)} - \mathbf{x}_0'^{(p)}) = 0, \quad \text{for } \mathbf{x}'^{(p)} \in \Omega'_p, \tag{5.10}$$

entirely similar to that for an isotropic layered medium if one chooses $\alpha_1 = (\kappa_n^{(1)}/\kappa_2^{(1)})^{\frac{1}{n-2}}$. Next, the interface continuity conditions (5.2) become

$$G^p|_{J'_p} = G^{p+1}|_{J'_p}, \quad \kappa_2^{(p)} \frac{\partial G^p}{\partial x_1'^{(p)}}\bigg|_{J'_p} = \kappa_2^{(p+1)} \frac{\partial G^{p+1}}{\partial x_1'^{(p+1)}}\bigg|_{J'_p}. \tag{5.11}$$

The transformed governing system (5.10) and (5.11) is exactly the same as that of an n-dimensional Green's function of a multilayered medium with isotropic constituents. If the isotropic Green's function for a layered medium can be known a priori, then the anisotropic Green's function for a layered medium, fulfilling the constraints (5.7), can be readily obtained without further derivations. We emphasize again that the exact linkage between the anisotropic and isotropic layered media and bimaterials depends crucially on whether the conductivities of the constituent layers satisfy the constraint conditions (5.7).

To further elaborate the conditions, we focus more on the common situations for two- and three-dimensional spaces. For a two-dimensional case, (5.7) simply follows

$$\alpha_1 \frac{\kappa_1^{(1)}}{\kappa_2^{(1)}} = \alpha_2 \frac{\kappa_1^{(2)}}{\kappa_2^{(2)}} = \cdots = \alpha_m \frac{\kappa_1^{(m)}}{\kappa_2^{(m)}}. \tag{5.12}$$

This suggests that one can select the value of α_p as

$$\frac{\alpha_p}{\alpha_1} = \frac{\frac{\kappa_1^{(1)}}{\kappa_2^{(1)}}}{\frac{\kappa_1^{(p)}}{\kappa_2^{(p)}}}, \quad p = 2,\ldots,m, \tag{5.13}$$

so that (5.12) is identically satisfied. In other words, the linkages between the Green's functions of anisotropic and isotropic layered media can always be proven, without regarding to the values of phase conductivities. We mention that a similar method was adopted by Ma and Chang (2004) for a two-dimensional layered medium with a focus on the integral transform solution for the isotropic Green's functions for a layered medium. Here our framework is more general for an arbitrary dimensional space.

For a three-dimensional space, the condition (5.7) becomes

$$\begin{aligned}
&\alpha_1 \frac{m_2^{22(1)}}{\kappa_1^{(1)}\kappa_2^{(1)}} = \alpha_2 \frac{m_2^{22(2)}}{\kappa_1^{(2)}\kappa_2^{(2)}} = \alpha_3 \frac{m_2^{22(3)}}{\kappa_1^{(3)}\kappa_2^{(3)}} = \cdots = \alpha_m \frac{m_2^{22(m)}}{\kappa_1^{(m)}\kappa_2^{(m)}}, \\
&\alpha_1 \frac{m_3^{32(1)}}{\kappa_2^{(1)}\kappa_3^{(1)}} = \alpha_2 \frac{m_3^{32(2)}}{\kappa_2^{(2)}\kappa_3^{(2)}} = \alpha_3 \frac{m_3^{32(3)}}{\kappa_2^{(3)}\kappa_3^{(3)}} = \cdots = \alpha_m \frac{m_3^{32(m)}}{\kappa_2^{(m)}\kappa_3^{(m)}}, \\
&\alpha_1 \frac{m_3^{33(1)}}{\kappa_2^{(1)}\kappa_3^{(1)}} = \alpha_2 \frac{m_3^{33(2)}}{\kappa_2^{(2)}\kappa_3^{(2)}} = \alpha_3 \frac{m_3^{33(3)}}{\kappa_2^{(3)}\kappa_3^{(3)}} = \cdots = \alpha_m \frac{m_3^{33(m)}}{\kappa_2^{(m)}\kappa_3^{(m)}},
\end{aligned} \tag{5.14}$$

which implies that the phase conductivities must fulfill the constraints

$$\begin{aligned}
\frac{\kappa_1^{(1)}}{\kappa_2^{(1)}} : \frac{\kappa_1^{(2)}}{\kappa_2^{(2)}} : \frac{\kappa_1^{(3)}}{\kappa_2^{(3)}} : \cdots : \frac{\kappa_1^{(m)}}{\kappa_2^{(m)}} &= \frac{m_3^{32(1)}}{\kappa_2^{(1)}\kappa_3^{(1)}} : \frac{m_3^{32(2)}}{\kappa_2^{(2)}\kappa_3^{(2)}} : \frac{m_3^{32(3)}}{\kappa_2^{(3)}\kappa_3^{(3)}} : \cdots : \frac{m_3^{32(m)}}{\kappa_2^{(m)}\kappa_3^{(m)}} \\
&= \frac{\kappa_2^{(1)}}{\kappa_3^{(1)}} : \frac{\kappa_2^{(2)}}{\kappa_3^{(2)}} : \frac{\kappa_2^{(3)}}{\kappa_3^{(3)}} : \cdots : \frac{\kappa_2^{(m)}}{\kappa_3^{(m)}} = \frac{1}{\alpha_1} : \frac{1}{\alpha_2} : \frac{1}{\alpha_3} : \cdots : \frac{1}{\alpha_m}.
\end{aligned} \tag{5.15}$$

Note that we have used the identities $m_2^{22} = \kappa_1^2$ and $m_3^{33} = \kappa_2^2$ to simplify the relations. Apparently, this relation will not be automatically fulfilled for any given three sets of conductivity tensors and thus (5.15) poses a restriction for the applicability of the exact linkage. To our knowledge, the linkages for the three- and higher-dimensional layered spaces are new. Yet, we are not aware of the existence of explicit solutions for the Green's function of an isotropic layered medium for three- and higher-dimensional spaces. As the latter boundary valued problems are indeed quite complicated, it is not our objective to resolve the isotropic Green's functions for an n-dimensional layered medium in this work. We simply point out that the anisotropic Green's functions can be obtained in principle, if the isotropic one can be known a priori and the constraints (5.15) are fulfilled.

6. Concluding remarks

We have shown that the n-dimensional anisotropic Green functions in conductions for an unbounded space and a half-space can always be obtained from those of the isotropic ones. For bimaterials and multilayered spaces, the exact links can also be found for two-dimensional planes. But for three- and higher-dimensional spaces, the linkages can only be proven if the constituent conductivities satisfy further algebraic constraints. Of course, the latter restriction does not necessarily mean that the corresponding Green's functions do not exist; it simply reflects the fact that there is no such simple linkage between the two configurations. We note that the present affine coordinate formulation, incorporating with a specific transformation matrix relevant to the Cholesky decomposition, offers a simple feature in dealing with various boundary valued problems with straight boundaries. This is clearly a major strength of the method. It may seem plausible that similar problems for cylindrically or spherically anisotropic solids with circular or spherical boundaries or interfaces can be looked into in future studies. We finally remark that the affine transformation method can also be applied to resolve the Green's function for an unbounded exponentially graded solid (Kuo and Chen, 2005). But, in general, the applicability of the method to a generally functionally graded solid needs to be studied case by case.

Acknowledgement

This work was supported by the National Science Council, Taiwan, under contract NSC 93-2211-E006-005.

Appendix A. How to find the transformation matrix A

In this Appendix, we discuss the explicit forms for the matrix **A** that satisfies the condition (2.5). To do this, we first recall the Cholesky decomposition theorem in matrix theory (Horn and Johnson, 1985): any $n \times n$ positive-definite symmetric matrix **M** can be uniquely factorized into a product $\mathbf{M} = \mathbf{B}\mathbf{B}^{\mathrm{T}}$, where **B** is a positive-definite lower triangular $n \times n$ matrix. Now back to (2.5), with the decomposition theorem, if one sets $\mathbf{M} = \alpha^{-2}\mathbf{k}_n$, then the transformation matrix **A** can be determined as

$$\mathbf{A} = \mathbf{B}^{-1}_{\text{Cholesky}}. \tag{A.1}$$

Note that **B** is a positive-definite lower triangular matrix, and so is **A**. We mention that the method of Cholesky decomposition is originally used to provide a systematic process for solving a linear system in which the coefficient matrix is symmetric and positive-definite. An iterative procedure for Cholesky decomposition is outlined in Horn and Johnson (1985). Specifically, the transformation matrix **A** can be explicitly written as in (4.10) for a two-dimensional plane, (4.18) for a three-dimensional space, and (4.24) for an n-dimensional space. We mention that, for $n = 2$ and 3, the above expressions for **A** can also be derived from some symbolic algebra, e.g., MAPLE. For a higher-dimensional space, the expression (4.24) was made by a simple conjecture and has been correctly verified.

Appendix B. Green's functions for an isotropic half-space

Green's functions in conduction for an isotropic half-space (2.7) with boundary conditions (3.5) are recorded in the Appendix. To distinguish the **x**-coordinate employed in the main text for a general anisotropic space, here we use **y** as the referenced (isotropic) coordinate. The material is isotropic in which its conductivity tensor is denoted by $k\mathbf{I}_n$, where $\mathbf{I}_n$ is a unit $(n \times n)$ matrix. We assume that a point heat source is prescribed at a certain point $\mathbf{y}_0$. For a two-dimensional infinite plane, the Green's function is known as (see for instance, Greenberg, 1971, p. 64, p. 81, p. 86 and p. 91)

$$G^{\infty}_{2\mathrm{D}}(\mathbf{y};\mathbf{y}_0) = -\frac{\log|\mathbf{y}-\mathbf{y}_0|}{2\pi k}. \tag{B.1}$$

For a half-plane which occupies the region $0 \leqslant y_1 < \infty$, the Green's functions are

$$G^{\mathrm{I,II}}_{2\mathrm{D}}(\mathbf{y};\mathbf{y}_0) = -\frac{\log|\mathbf{y}-\mathbf{y}_0|}{2\pi k} \pm \frac{\log|\mathbf{y}-\mathbf{y}^i_0|}{2\pi k} \tag{B.2}$$

for the first- and second-kind of homogeneous boundary condition, and

$$G^{\mathrm{III}}_{2\mathrm{D}}(\mathbf{y};\mathbf{y}_0) = -\frac{\log|\mathbf{y}-\mathbf{y}_0|}{2\pi k} - \frac{\log|\mathbf{y}-\mathbf{y}^i_0|}{2\pi k} - \int_{-\infty}^{-y^0_1} \tilde{h}(\eta)\frac{\log|\mathbf{y}-\mathbf{y}^i_\eta|}{2\pi k}\,\mathrm{d}\eta \tag{B.3}$$

for the third-kind boundary condition, where

$$\mathbf{y}^i_0 = (-y^0_1, y^0_2)^{\mathrm{T}}, \quad \mathbf{y}^i_\eta = (\eta, y^0_2)^{\mathrm{T}}, \quad \tilde{h}(\eta) = -2\tilde{p}\exp(\tilde{p}(y^0_1+\eta)). \tag{B.4}$$

This means that $\mathbf{y}_0^i$ is the image point of the source point $\mathbf{y}_0$ relevant to the plane $y_1 = 0$. For convenience, we may write $\mathbf{y}_0^i = \mathbf{R}_2\mathbf{y}_0$, where $\mathbf{R}_2$ is a (2×2) reflectional matrix with respect to $y_1 = 0$, given by

$$\mathbf{R}_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{diag}(-1, 1). \tag{B.5}$$

For an n-dimensional space ($n \geqslant 3$), the fundamental solution for Laplace's operator can be found in John (1982, p. 97)

$$G_{n\mathrm{D}}^{\infty}(\mathbf{y};\mathbf{y}_0) = \frac{|\,\mathbf{y}-\mathbf{y}_0|^{2-n}}{(n-2)\omega_n k}, \quad \omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \quad \text{for } n \geqslant 3, \tag{B.6}$$

where ω_n denotes the surface area of the unit sphere in $\mathbb{R}^n$. By the method of image, the Green's function for the first- and second-kind boundary conditions are

$$G_{n\mathrm{D}}^{\mathrm{I,II}}(\mathbf{y};\mathbf{y}_0) = \frac{|\,\mathbf{y}-\mathbf{y}_0|^{2-n}}{(n-2)\omega_n k} \mp \frac{|\,\mathbf{y}-\mathbf{y}_0^i|^{2-n}}{(n-2)\omega_n k}, \tag{B.7}$$

where

$$\mathbf{y}_0^i = (-y_1^0, y_2^0, \ldots, y_n^0)^{\mathrm{T}}, \quad \text{or equivalently} \quad \mathbf{y}_0^i = \mathbf{R}_n\mathbf{y}_0, \tag{B.8}$$

and $\mathbf{R}_n$ is the $(n\times n)$ diagonal matrix defined by diag$(-1, 1, \ldots, 1)$.
For the third-kind boundary condition (3.5), the Green's function can be expressed in the following modified image form (Greenberg, 1971, p. 86)

$$G_{n\mathrm{D}}^{\mathrm{III}}(\mathbf{y};\mathbf{y}_0) = \frac{|\,\mathbf{y}-\mathbf{y}_0|^{2-n}}{(n-2)\omega_n k} + \frac{|\,\mathbf{y}-\mathbf{y}_0^i|^{2-n}}{(n-2)\omega_n k} + \int_{-\infty}^{-y_1^0} \tilde{h}(\eta)\frac{|\,\mathbf{y}-\mathbf{y}_\eta^i|^{2-n}}{(n-2)\omega_n k}\,\mathrm{d}\eta, \tag{B.9}$$

where $\tilde{h}(\eta)$ is defined as (B.4)$_3$ and

$$\mathbf{y}_\eta^i = (\eta, y_2^0, \ldots, y_n^0)^{\mathrm{T}}. \tag{B.10}$$

Appendix C. Green's functions for an isotropic bimaterial

In this Appendix we give the Green's functions for isotropic bimaterials. The bimaterial is composed of two anisotropic half-spaces bonded along $y_1 = 0$ in an n-dimensional space. The material A occupies the region $y_1 > 0$ and material B the remaining half-space $y_1 < 0$. We suppose that a point heat source is prescribed at the point $\mathbf{y}_0$ inside the region A. The Green's functions can be obtained analytically as

$$G_{2\mathrm{D}}^{A}(\mathbf{y};\mathbf{y}_0) = -\frac{\log|\,\mathbf{y}-\mathbf{y}_0\,|}{2\pi k^A} - \frac{\log|\,\mathbf{y}-\mathbf{y}_0^i\,|}{2\pi k^A}\left(\frac{k^A-k^B}{k^A+k^B}\right), \tag{C.1}$$

$$G_{2\mathrm{D}}^{B}(\mathbf{y};\mathbf{y}_0) = -\frac{\log|\,\mathbf{y}-\mathbf{y}_0\,|}{\pi(k^A+k^B)}, \tag{C.2}$$

for a two-dimensional plane, where $\mathbf{y}_0^i$ has been defined in (B.4), and

$$G_{n\mathrm{D}}^{A}(\mathbf{y};\mathbf{y}_0) = \frac{|\,\mathbf{y}-\mathbf{y}_0|^{2-n}}{(n-2)\omega_n k^A} + \frac{k^A-k^B}{k^A+k^B}\,\frac{|\,\mathbf{y}-\mathbf{y}_0^i|^{2-n}}{(n-2)\omega_n k^A}, \tag{C.3}$$

and

$$G_{n\mathrm{D}}^{B}(\mathbf{y};\mathbf{y}_0)=\frac{2k^B}{k^A+k^B}\frac{|\,\mathbf{y}-\mathbf{y}_0|^{2-n}}{(n-2)\omega_n k^B}, \tag{C.4}$$

for an n-dimensional space, where $n \geqslant 3$, and $\mathbf{y}_0^i$ has been defined in (B.8).

References

Berger, J.R., Skilowitz, J.L., Tewary, V.K., 2000. Green's function for steady-state heat conduction in a bimaterial composite solid. Comput. Mech. 25, 627–633.

Carslaw, H.S., Jaeger, J.C., 1959. Conduction of Heat in Solids. Clarendon Press, Oxford.

Chang, Y.P., 1977. Analytic solution for heat conduction in anisotropic media in infinite, semi-infinite, and two-plane-bounded regions. Int. J. Heat Mass Transfer 20, 1019–1028.

Chen, T., 1993. Green's functions and the non-uniform transformation problem in a piezoelectric medium. Mech. Res. Commun. 20, 271–278.

DeSanto, J.A., 1992. Scalar Wave Theory—Green's Functions and Applications. Springer-Verlag, New York.

Greenberg, M.D., 1971. Application of Green's Functions in Science and Engineering. Prentice-Hall, New Jersey.

Horgan, C.O., Miller, K.L., 1994. Antiplane shear deformations for homogeneous and inhomogeneous anisotropic linearly elastic solids. J. Appl. Mech. 61, 23–29.

Horn, R.A., Johnson, C.R., 1985. Matrix Analysis. Cambridge University Press, Cambridge.

John, F., 1982. Partial Differential Equations. Springer-Verlag, New York.

Kuo, H.-Y., Chen, T., 2005. Steady and transient Green's functions for anisotropic conduction in an exponentially graded solid. Int. J. Solids Struct. 42, 1111–1128.

Ma, C.C., Chang, S.W., 2004. Analytical exact solutions of heat conduction problems for anisotropic multi-layered media. Int. J. Heat Mass Transfer 47, 1643–1655.

Milton, G.W., 2002. The Theory of Composites. Cambridge University Press, Cambridge.

Norris, A.N., 1994. Dynamic Green's functions in anisotropic piezoelectric, thermoelastic and poroelastic solids. Proc. R. Soc. London A 447, 175–188.

Özişik, M.N., 1993. Heat Conduction, second ed. John Wiley & Sons, Inc., New York.

Sokolnikoff, I.S., 1956. Mathematical Theory of Elasticity. McGraw-Hill, New York.

Ting, T.C.T., 1996. Anisotropic Elasticity: Theory and Applications. Oxford University Press, Oxford.